

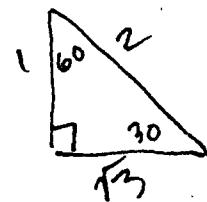
## Compound Angle Formulae

### Investigation I: Addition and Subtraction Formulas for Sine

1. Evaluate the following:

a)  $\sin \frac{\pi}{2} - 90^\circ$   
 $= \sin(90^\circ)$   
 $= 1$

b)  $\sin \frac{\pi}{3} \cos \frac{\pi}{6} + \cos \frac{\pi}{3} \sin \frac{\pi}{6}$   
 $= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$   
 $= \frac{3}{4} + \frac{1}{4}$   
 $= 1$



c) What do you notice about the results? answers are the same

d) What do you notice about the angles used?  $\frac{\pi}{2} = \pi/3 + \pi/6$

Conclusion:  $\sin(a+b) = \sin a \cos b + \cos a \sin b$

2. Evaluate the following:

a)  $\sin \frac{\pi}{3} - 60^\circ$

$= \frac{\sqrt{3}}{2}$

b)  $\sin \frac{\pi}{2} \cos \frac{\pi}{6} - \cos \frac{\pi}{2} \sin \frac{\pi}{6}$

$= (1)\left(\frac{\sqrt{3}}{2}\right) - (0)\left(\frac{1}{2}\right)$

$= \frac{\sqrt{3}}{2}$

c) What do you notice about the results? answers are the same

d) What do you notice about the angles used?  $\pi/3 = \pi/2 - \pi/6$

Conclusion:  $\sin(a-b) = \sin a \cos b - \cos a \sin b$

Addition Formula for Sine	Subtraction Formula for Sine
$\sin(a+b) = \sin a \cos b + \cos a \sin b$	$\sin(a-b) = \sin a \cos b - \cos a \sin b$

Example 1: Find the exact value of  $\sin \frac{\pi}{12}$ .  $\swarrow 15^\circ$

Strategy: Describe  $\frac{\pi}{12}$  as the sum or difference of standard angles we have worked with (e.g.,  $\frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{2}$ , and multiples of these).

$$\begin{aligned}
 & \sin(15^\circ) \\
 &= \sin(60^\circ - 45^\circ) \\
 &= \sin 60 \cos 45 - \cos 60 \sin 45 \\
 &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\
 &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}
 \end{aligned}$$

### Investigation II: Addition and Subtraction Formulas for Cosine

1. Evaluate the following.

a)  $\cos \frac{\pi}{2} \swarrow 90^\circ$

$= 0$

b)  $\cos \frac{\pi}{3} \cos \frac{\pi}{6} - \sin \frac{\pi}{3} \sin \frac{\pi}{6}$

$$\begin{aligned}
 &= \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right) \\
 &= \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4} \\
 &= 0
 \end{aligned}$$

c) What do you notice about the results? Same

d) What do you notice about the angles used?  $\frac{\pi}{2} = \frac{\pi}{3} + \frac{\pi}{6}$

Conclusion:  $\cos(a+b) = \cos a \cos b - \sin a \sin b$

2. Evaluate the following.

a)  $\cos \frac{\pi}{3} \swarrow 60^\circ$

$= \frac{1}{2}$

b)  $\cos \frac{\pi}{2} \cos \frac{\pi}{6} + \sin \frac{\pi}{2} \sin \frac{\pi}{6}$

$$\begin{aligned}
 &= 0 \left(\frac{\sqrt{3}}{2}\right) + 1 \left(\frac{1}{2}\right) \\
 &= \frac{1}{2}
 \end{aligned}$$

c) What do you notice about the results? the same

d) What do you notice about the angles used?  $\frac{\pi}{3} = \frac{\pi}{2} - \frac{\pi}{6}$

Conclusion:  $\cos(a-b) = \cos a \cos b + \sin a \sin b$

Addition Formula for Cosine	Subtraction Formula for Cosine
$\cos(a + b) = \cos a \cos b - \sin a \sin b$	$\cos(a - b) = \cos a \cos b + \sin a \sin b$

Example 2: Find the exact value of  $\cos \frac{5\pi}{12}$

$$\begin{aligned}
 & \cos(75^\circ) \\
 &= \cos(45^\circ + 30^\circ) \\
 &= \cos 45 \cos 30 - \sin 45 \sin 30 \\
 &= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\
 &= \frac{\sqrt{6} - \sqrt{2}}{4}
 \end{aligned}$$

### Addition and Subtraction Formulas for Tangent

Addition Formula for Tangent	Subtraction Formula for Tangent
$\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$	$\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$

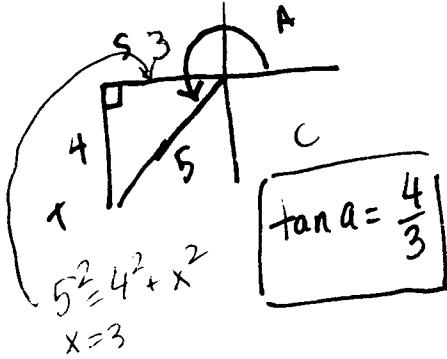
Prove the Addition Formula for Tangent using the identity  $\tan x = \frac{\sin x}{\cos x}$ .

(The Subtraction Formula for Tangent will be proven in the homework.)

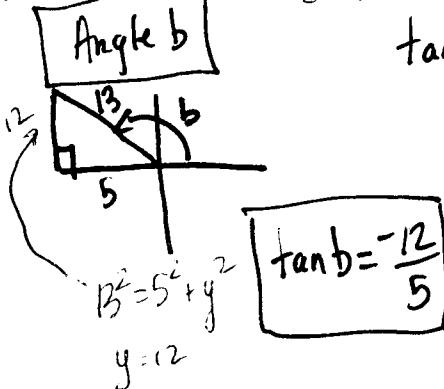
**Example 3:** If  $\sin a = -\frac{4}{5}$ ,  $\pi \leq a \leq \frac{3\pi}{2}$  and  $\cos b = -\frac{5}{13}$ ,  $\frac{\pi}{2} \leq b \leq \pi$ , evaluate  $\tan(a+b)$ .

Strategy: In order to use the sum formula for tangent, the tangent values for each angle must be found.

**Angle a**



**Angle b**



$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

$$\begin{aligned} &= \frac{\frac{4}{3} - \frac{12}{5}}{1 - (\frac{4}{3})(-\frac{12}{5})} \\ &= \frac{-\frac{16}{15}}{\frac{63}{15}} = \frac{-16}{63} \\ &= -\frac{16}{63} \end{aligned}$$

## Homework

### Knowledge

1. Express as a single trigonometric function.

a)  $\cos 2a \cos a - \sin 2a \sin a$   
 $= \cos(3a)$

b)  $\cos x \cos 4x + \sin x \sin 4x$

c)  $\sin 2m \cos m + \cos 2m \sin m$

d)  $\frac{\tan 2a + \tan 3a}{1 - \tan 2a \tan 3a}$

e)  $\frac{\tan 7 - \tan 9}{1 + \tan 7 \tan 9}$

2. Find the exact value of each of the following.

a)  $\sin\left(\frac{\pi}{4} - \frac{\pi}{3}\right)$

b)  $\cos\left(-\frac{\pi}{6} - \frac{\pi}{4}\right)$

c)  $\tan\left(-\frac{3\pi}{4} + \frac{2\pi}{3}\right)$

d)  $\cos\frac{\pi}{7} \cos\frac{4\pi}{21} - \sin\frac{\pi}{7} \sin\frac{4\pi}{21}$

e)  $\sin\frac{5\pi}{36} \cos\frac{5\pi}{18} + \cos\frac{5\pi}{36} \sin\frac{5\pi}{18}$  (Application)

### Application

3. Evaluate using the formulae developed in this lesson.

a)  $\sin\frac{11\pi}{12}$

b)  $\cos\frac{13\pi}{12}$

c)  $\tan\frac{7\pi}{12}$

d)  $\tan\frac{5\pi}{12}$

4. If  $\cos x = -\frac{5}{13}$ ,  $x$  is in the interval  $\left(\frac{\pi}{2}, \pi\right)$  and  $\tan y = \frac{4}{3}$ ,  $y$  is in the interval  $\left(\pi, \frac{3\pi}{2}\right)$ , evaluate each of the following.

a)  $\sin(x - y)$       b)  $\sin(x + y)$       c)  $\cos(x - y)$       d)  $\tan(x + y)$

5. If  $\sin x = -\frac{1}{3}$ ,  $\pi < x < \frac{3\pi}{2}$  and  $\cos y = \frac{2}{5}$ ,  $\frac{3\pi}{2} < y < 2\pi$ , find the value of  $\sec(x - y)$ .

### Thinking

6. Prove the Subtraction Formula for Tangent using the identity  $\tan x = \frac{\sin x}{\cos x}$ .

7. Find the **double angle formulas** using the sum and difference formulas and simplifying.

a)  $\sin 2x = \sin(x + x)$       b)  $\cos 2x = \cos(x + x)$       c)  $\tan 2x = \tan(x + x)$

8. If  $\cos x = -\frac{4}{5}$ ,  $\frac{\pi}{2} < x < \pi$ , find the value of  $\sin 2x$  and  $\cos 2x$ . Determine the quadrant of the angle  $2x$ .

9. If  $\sin x = \frac{12}{13}$ ,  $0 < x < \frac{\pi}{2}$ , evaluate  $\sin 2x$  and  $\cos 2x$ . Determine the quadrant of the angle  $2x$ .

### Answers:

1a)  $\cos 3a$  b)  $\cos(-3x)$  or  $\cos 3x$  c)  $\sin 3m$  d)  $\tan 5a$  e)  $\tan(-2)$

2a)  $\frac{1-\sqrt{3}}{2\sqrt{2}}$  b)  $\frac{\sqrt{3}-1}{2\sqrt{2}}$  c)  $\frac{1-\sqrt{3}}{1+\sqrt{3}}$  d)  $\frac{1}{2}$  e)  $\frac{1+\sqrt{3}}{2\sqrt{2}}$

3a)  $\frac{\sqrt{3}-1}{2\sqrt{2}}$  b)  $\frac{-1-\sqrt{3}}{2\sqrt{2}}$  c)  $\frac{1+\sqrt{3}}{1-\sqrt{3}}$  d)  $\frac{\sqrt{3}+1}{1-\sqrt{3}}$

4a)  $-\frac{56}{65}$  b)  $-\frac{16}{65}$  c)  $-\frac{33}{65}$  d)  $-\frac{16}{63}$  e)  $\frac{15}{\sqrt{21-4\sqrt{2}}}$

7a)  $\sin 2x = 2 \sin x \cos x$  b)  $\cos 2x = \cos^2 x - \sin^2 x$  c)  $\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$

8)  $-\frac{24}{25}, \frac{7}{25}$ , 4<sup>th</sup> quadrant

9)  $\frac{120}{169}, -\frac{119}{169}$ , 2<sup>nd</sup> quadrant